

On the Relation between two Rotation Metrics

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Abstract In their work "Global Optimization through Rotation Space Search" [5], Richard Hartley and Fredrik Kahl introduce a global optimization strategy for problems in geometric computer vision, based on rotation space search using a branch-and-bound algorithm. In its core, Lemma 2 of their publication is the important foundation for a class of global optimization algorithms, which is adopted over a wide range of problems in subsequent publications. This lemma relates a metric on rotations represented by rotation matrices with a metric on rotations in axis-angle representation. This work focuses on a proof for this relationship, which is based on Rodrigues' Rotation Theorem for the composition of rotations in axis-angle representation [9, 1].

1 Introduction

In geometry, various representations exist to describe a rotation in Euclidean 3-space. The focus of this work is the relationship between two metrics on the following two rotation representations.

Rotation Matrices The group of isometric, linear transformations which preserve handedness in space is referred to as $\text{SO}(3) := \{R \in \mathbb{R}^{3 \times 3} | R^T R = I, \det R = 1\}$.

Multiplied Axis-Angle $\mathcal{R} \subset \mathbb{R}^3$ denotes the space of multiplied axis-angle¹ representations $\mathbf{r} = \alpha \mathbf{a}$ of rotations with angle $\alpha \in [0, \pi]$ about the axis $\mathbf{a} \in \mathbb{R}^3$ with $\|\mathbf{a}\| = 1$. It describes a closed ball of radius π in \mathbb{R}^3 .

Geometric computer vision or reconstruction problems are often formulated as a task of minimizing a cost or objective function. In general, these objective functions are non-convex. As a result, standard local optimization algorithms only yield locally optimal results. Hartley and Kahl contributed a global optimization strategy for problems in geometric computer vision, based on rotation space search using a branch-and-bound algorithm [5]. Their work is the foundation for several subsequent contributions [2, 3, 4, 6, 7, 10, 11]. Hartley and Kahl's key contribution, which enables the branch-and-bound search strategy, is Lemma 1. It relates a metric on rotations represented by rotation matrices from $\text{SO}(3)$ with a metric on rotations in multiplied axis-angle representation \mathcal{R} .

¹Also known as Rodrigues parameters, named after Benjamin Olinde Rodrigues (1795-1851)

2 Previous Work

This section restates definitions and lemmas from [5]. Definition 1 introduces a metric on the space $\text{SO}(3)$ of rotation matrices.

Definition 1 Let the rotation matrices $R_A, R_B \in \text{SO}(3)$ represent two rotations in Euclidean 3-space. The operation $d_{\angle} : \text{SO}(3) \times \text{SO}(3) \rightarrow [0, \pi]$ defines the metric

$$d_{\angle}(R_A, R_B) := \angle(R_B^{-1} R_A), \quad (1)$$

where the angle operator $\angle(\cdot)$ yields the rotation angle of the given rotation matrix after decomposition into rotation axis and angle.

Lemma 1 relates this metric to the Euclidean distance of the respective rotations in multiplied axis-angle representation.

Lemma 1 Let two rotations about axes $\mathbf{a}_A, \mathbf{a}_B \in \mathbb{R}^3$, $\|\mathbf{a}_A\| = \|\mathbf{a}_B\| = 1$ by the angles $\alpha, \beta \in [0, \pi]$ be represented by the multiplied axis-angle representations $\mathbf{r}_A = \alpha \mathbf{a}_A \in \mathcal{R}$ and $\mathbf{r}_B = \beta \mathbf{a}_B \in \mathcal{R}$ as well as the rotation matrices $R_A, R_B \in \text{SO}(3)$. The following relationship holds:

$$d_{\angle}(R_A, R_B) \leq \|\mathbf{r}_A - \mathbf{r}_B\|. \quad (2)$$

This work focuses on a proof for Lemma 1.

3 The Proof

The key idea of this proof is to cast Lemma 1 to an upper bound on the rotation angle of the composed rotation $R_B R_A$. To achieve this, all representations of rotation B are inverted without loss of generality. B 's multiplied axis-angle representation \mathbf{r}_B is substituted by $-\mathbf{r}_B$ and its rotation matrix representation R_B by R_B^{-1} , respectively:

$$d_{\angle}(R_A, R_B^{-1}) \leq \|\mathbf{r}_A - (-\mathbf{r}_B)\|. \quad (3)$$

The verification of this upper bound is grouped into three sections. Section 3.1 reformulates the left hand side fully in terms of the three angles α, β and φ , where φ denotes the angle enclosed by \mathbf{a}_A and \mathbf{a}_B

$$\varphi = \arccos \mathbf{a}_A^T \mathbf{a}_B. \quad (4)$$

Section 3.2 follows the same goal for the right hand side. Section 3.3 then verifies inequality (3) by reducing it to a test for function convexity.

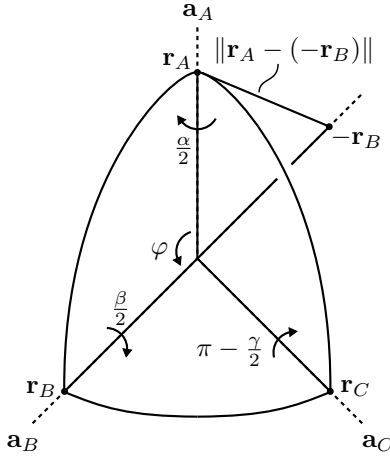


Figure 1: Geometrical illustration of Rodrigues' rotation theorem on the composition of rotations in axis-angle representation [1, 9]. The theorem studies the spherical triangle spanned by the rotation axes \mathbf{a}_A , \mathbf{a}_B and \mathbf{a}_C . In this triangle, \mathbf{a}_C and γ are the axis and the angle of the composed rotation $B \circ A$. The three corresponding rotation angles α , β and γ appear as half-angles.

3.1 The Left Hand Side

Applying Definition 1 to the rotation matrix metric d_{\angle} on the left hand side of (3) yields

$$d_{\angle}(\mathbf{R}_A, \mathbf{R}_B^{-1}) = \angle(\mathbf{R}_B \mathbf{R}_A), \quad (5)$$

the angle of the composed rotation $B \circ A$. The goal of this section is to express this composition directly in \mathcal{R} . This is enabled by Rodrigues' Rotation Theorem on the composition of rotations in axis-angle representation [1, 9]. Figure 1 illustrates its geometrical interpretation. The theorem provides a closed form solution for the angle

$$\begin{aligned} \angle(\mathbf{R}_B \mathbf{R}_A) \\ = 2 \arccos \left(\cos \frac{\alpha}{2} \cos \frac{\beta}{2} - \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \varphi \right). \end{aligned} \quad (6)$$

By substituting $1 - 2d = \cos \varphi$, $\alpha' = \frac{\alpha}{2}$ and $\beta' = \frac{\beta}{2}$, the argument of the arccos in (6) expands to

$$\cos \alpha' \cos \beta' - (1 - 2d) \sin \alpha' \sin \beta' \quad (7)$$

$$= \cos \alpha' \cos \beta' - \sin \alpha' \sin \beta' \quad (8)$$

$$+ d \sin \alpha' \sin \beta' + d \sin \alpha' \sin \beta' \\ = \cos \alpha' \cos \beta' - \sin \alpha' \sin \beta' \quad (9)$$

$$+ d \sin \alpha' \sin \beta' + d \sin \alpha' \sin \beta' \\ + d \cos \alpha' \cos \beta' - d \cos \alpha' \cos \beta'.$$

Applying the trigonometric addition and subtraction theorem $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$ simplifies this argument to

$$\begin{aligned} d \cos(\alpha' - \beta') - d \cos(\alpha' + \beta') + \cos(\alpha' + \beta') \\ = d \cos(\alpha' - \beta') + (1 - d) \cos(\alpha' + \beta'). \end{aligned} \quad (10)$$

In summary, for the left hand side of (3) holds

$$\begin{aligned} \angle(\mathbf{R}_B \mathbf{R}_A) \\ = 2 \arccos(d \cos(\alpha' - \beta') + (1 - d) \cos(\alpha' + \beta')). \end{aligned} \quad (11)$$

3.2 The Right Hand Side

Figure 1 illustrates the expression $\|\mathbf{r}_A - (-\mathbf{r}_B)\|$, on the right hand side of (3). When constructing the triangle spanned by the origin $\mathbf{0}$, \mathbf{r}_A and $-\mathbf{r}_B$, the length of two sides ($\|\mathbf{r}_A\|$ and $\|-\mathbf{r}_B\|$) and the angle enclosed by these two sides ($\pi - \varphi$) is known. The law of cosines yields the length of the remaining side

$$\begin{aligned} \|\mathbf{r}_A - (-\mathbf{r}_B)\| \\ = \sqrt{\|\mathbf{r}_A\|^2 - 2 \|\mathbf{r}_A\| \|\mathbf{r}_B\| \cos(\pi - \varphi) + \|\mathbf{r}_B\|^2}. \end{aligned} \quad (12)$$

By definition of \mathbf{r}_A and \mathbf{r}_B and the symmetry of the cosine, this is equal to

$$2 \sqrt{\left(\frac{\alpha}{2}\right)^2 - 2 \frac{\alpha}{2} \frac{\beta}{2} (-\cos \varphi) + \left(\frac{\beta}{2}\right)^2}. \quad (13)$$

Again substituting $1 - 2d = \cos \varphi$, $\alpha' = \frac{\alpha}{2}$ and $\beta' = \frac{\beta}{2}$ yields for the argument of the square root

$$\alpha'^2 - 2\alpha'\beta'(2d - 1) + \beta'^2 \quad (14)$$

$$= \alpha'^2 - 2d\alpha'\beta' - 2d\alpha'\beta' + 2\alpha'\beta' + \beta'^2 \quad (15)$$

$$= \alpha'^2 - 2d\alpha'\beta' - 2d\alpha'\beta' + 2\alpha'\beta' + \beta'^2 \\ + d\alpha'^2 - d\alpha'^2 + d\beta'^2 - d\beta'^2 \quad (16)$$

$$= d(\alpha'^2 - 2\alpha'\beta' + \beta'^2) \\ + (\alpha'^2 + 2\alpha'\beta' + \beta'^2) \\ - d(\alpha'^2 + 2\alpha'\beta' + \beta'^2) \quad (17)$$

$$= d(\alpha' - \beta')^2 + (1 - d)(\alpha' + \beta')^2. \quad (18)$$

In total, for the right hand side of (3) it holds

$$\begin{aligned} \|\mathbf{r}_A - (-\mathbf{r}_B)\| \\ = 2 \sqrt{d(\alpha' - \beta')^2 + (1 - d)(\alpha' + \beta')^2} \end{aligned} \quad (19)$$

3.3 Verifying the Inequality

Collecting both sides, (3) now is transformed to

$$\begin{aligned} \arccos(d \cos(\alpha' - \beta') + (1 - d) \cos(\alpha' + \beta')) \\ \leq \sqrt{d(\alpha' - \beta')^2 + (1 - d)(\alpha' + \beta')^2}. \end{aligned} \quad (20)$$

This relationship is strongly related to the property of relative convexity [8] and in the following is examined in a similar way. Since both arccos and the square root are non-negative functions, the inequality still holds for the square of both sides

$$\begin{aligned} \arccos^2(d \cos a + (1 - d) \cos b) \\ \leq d a^2 + (1 - d) b^2, \end{aligned} \quad (21)$$

where $a = \alpha' - \beta'$ and $b = \alpha' + \beta'$. Both α' and β' are limited to the interval $[0, \frac{\pi}{2}]$. The derived parameters a and b are thus within $[-\frac{\pi}{2}, \frac{\pi}{2}]$. On this interval, it is safe to substitute $a^2 = (\arccos \cos a)^2$ and $b^2 = (\arccos \cos b)^2$ to result in

$$\begin{aligned} \arccos^2(d \cos a + (1 - d) \cos b) \\ \leq d (\arccos \cos a)^2 + (1 - d) (\arccos \cos b)^2. \end{aligned} \quad (22)$$

Finally, simplifying the inequality by substituting $a' = \cos a$ and $b' = \cos b$ yields

$$\begin{aligned} & \arccos^2(d a' + (1-d) b') \\ & \leq d \arccos^2 a' + (1-d) \arccos^2 b'. \end{aligned} \quad (23)$$

This reduces the proof to verifying the convexity of the function \arccos^2 on the interval $[0, 1]$. For $f(x) = \arccos^2 x$ to be convex, its second derivative

$$f''(x) = \frac{2\sqrt{1-x^2} - 2x \arccos x}{(1-x^2)^{3/2}} \quad (24)$$

has to be non-negative. The denominator of f'' ranges in the interval $[0, 1]$. The first derivative of the numerator

$$\frac{\partial}{\partial x} 2\sqrt{1-x^2} - 2x \arccos x = -2 \arccos x \quad (25)$$

vanishes at $x = 1$, where the numerator of f'' assumes its minimum 0. \square

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